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Anomalous scaling of stock price dynamics within ARCH-models

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Abstract. We show that autoregressive-conditional-heteroskedasticity (ARCH) models can encompass the observed anomalous scaling properties of stock price dynamics remarkably well. We find that with a suitable choice of parameters, simple ARCH models can reproduce the non-standard scaling behavior of the central part of the probability distribution functions of stock prices at different time horizons, as empirically found for the Standard & Poors 500 (S&P 500) index data, but fail to reproduce the shape of the S&P 500 distribution, in particular at the smallest time horizon (1 min). A linear version of ARCH processes, denoted here as LARCH models, still preserving the anomalies observed, permits to fit the 1 min S&P 500 distribution more accurately.

PACS. 02.50.Ey Stochastic processes – 05.40.Fb Random walks and Levy flights – 87.23.Ge Dynamics of social systems – 89.90.+n Other topics in areas of applied and interdisciplinary physics

Rapid Note

Modelling the erratic evolution of stock prices by using the concept of random walks (RW) initiated about 100 years ago [1]. Nowadays, elaborated RW models describing scenarios of variable volatility such as autoregressive-conditional-heteroskedasticity (ARCH/GARCH) processes are widely used in finance [2–8]. In recent years a great advance in our empirical knowledge of the statistical properties of stock price variations has been achieved [9–16]; a prominent and intriguing feature is the anomalous scaling found near the central part of the associated leptokurtic distributions as a function of the time horizon [11], imposing a severe constraint to the existing RW models. It has been suggested that truncated Lévy flights can account for these observations [11], while the more traditional ARCH/GARCH processes are believed to fail in that goal [12].

In this Rapid Note we show, in contrast to the present belief, that ARCH processes can encompass the observed anomalous scaling of stock prices remarkably well, contributing to the extensively documented success of ARCH processes in the financial literature [2–8]. We illustrate our findings by considering the simplest model, an ARCH(1) process. A linear variant of the latter, denoted here as LARCH(1) process, is introduced in order to appropriately fit the real market data. This study suggests that ARCH-like processes may indeed contain some of the essential features responsible for the scaling anomalies observed in equity markets.

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We consider an uninterrupted stock market in which the trading transactions of an equity (or an index) are recorded trade-by-trade disregarding the time delay between two successive trading days due to overnight (or holiday) inactivity. For simplicity, the elapsed time τ between trades is assumed to be constant, typically of the order of few seconds. Once the trade-by-trade sequence has been obtained, we can derive the resulting behavior every m transactions, $m > 1$, corresponding to time horizons $\Delta t = m\tau$. For convenience, the stochastic model we consider is based on a discrete time random walk in a one-dimensional lattice of equidistant sites, lattice constant ℓ and lattice size L, corresponding to a fully discrete price dynamics.

The discrete coordinate i of the random walk represents the variation of the logarithm of the price Y_n of an equity at the nth transaction,

$$i_n \equiv \frac{1}{\ell}\left[\log Y_n - \log Y_{n-1}\right] \qquad (1)$$

where ℓ plays the role of a tick size for price returns and $-L < i_n < L$. In this case, the price Y_N after N transactions is given by $Y_N = Y_0 \exp(\sum_{n=1}^{N} i_n \ell)$. Since the results are qualitatively equivalent for different values of ℓ, we will assume $\ell = 1$ in what follows. We would like to note that effects of discrete quotes on the dynamics of stock prices have been studied recently, in which the obtained prices are rounded afterwards, down for the bid and up for the ask prices [17] (see also [18]).

The rules of the random walk are simple. Let a particle (or random walker) be at site i_n after the nth step. From site i_n, the particle can jump to another site j with a probability $W_{i_n \to j} \propto \exp(-V_{j,i_n})$, where the 'potential' V is assumed to have a parabolic shape, $V_{j,i_n} = j^2/(2\sigma_n^2)$, characterized by an ARCH(1) square 'volatility' (*cf.* Ref. [2]) $\sigma_n^2 = a + b\, i_n^2$ that depends on the lattice position of the particle at site i_n, where $a > 0$ and $b \geq 0$. At the $(n{+}1)$th step, the particle moves to i_{n+1} and a new potential $V_{j,i_{n+1}}$ takes place. Note that the minimum of the potential $V_{j,i}$ remains always at $j = 0$, while its *width*, σ_i, depends on i. The process can be simulated very efficiently since the transition probabilities $W_{i \to j} = \exp(-V_{j,i}) / \sum_{k=-L}^{L} \exp(-V_{k,i})$ need to be calculated only once. This is due to the Markovian character of the process. The new occupied site i_{n+1} is obtained in practice by taking a uniformly distributed random number, r, and choosing i_{n+1} such that it is the maximum value for which $\sum_{j=-L}^{i_{n+1}} W_{i_n \to j} \leq r$ holds.

The probability distribution function (PDF) for the discrete variations i, $P(i)$, of the process discussed above obeys the self-consistent relation, $P(i) = \sum_{j=-L}^{L} W_{j \to i}\, P(j)$, *i.e.*

$$P(i) = \sum_{j=-L}^{L} \frac{1}{S_j}\, \exp\left[-\frac{i^2}{2\sigma_j^2}\right] P(j) \qquad (2)$$

where $S_j \equiv \sum_{k=-L}^{L} \exp[-k^2/(2\sigma_j^2)]$, $\sigma_j^2 = a + b\, j^2$, and $P(i)$ is normalized such that $\sum_{i=-L}^{L} P(i) = 1$. The continuum analog to equation (2) corresponds to a PDF for continuous logarithmic price variations x [19]. For such standard ARCH(1) process, one has $\sigma_n^2 = a + b\, x_n^2$, with an average variance $\sigma^2 = a/(1-b)$. The resulting kurtosis, $\kappa \equiv \langle x^4 \rangle / \langle x^2 \rangle^2 = 3 + 6b^2/(1-3b^2)$, is finite for $b < 1/\sqrt{3} \cong 0.57735$ (see also [20] for other exact results obtained by mapping ARCH(1) models onto random multiplicative processes). In the discrete case, neither σ nor κ can be obtained in a closed form since in general their values depend explicitly on the self-consistent solution $P(i)$ in equation (2). We have verified numerically, however, that the condition $b < 1/\sqrt{3}$ yields a finite kurtosis for discrete processes too. Note that in the case of constant volatility, *i.e.* for $b = 0$, $P(i)$ reduces to a Gaussian and the prices Y_n are log-normally distributed.

It is believed that ARCH/GARCH processes can fit the PDF of stock prices quite well only at a given time horizon, missing the observed anomalous scaling at different time scales [11,14]. In what follows, we study the behavior of the ARCH(1) process at different time horizons from an initial tick-by-tick series. Specifically, we perform a trading simulation using the RW rules discussed above, and generate a long trade-by-trade sequence $\{i_n\}$ consisting of N points. Then, we record the logarithmic price changes every m trading transactions,

$$z_k \equiv \log Y_{km} - \log Y_{(k-1)m} \qquad (3)$$

i.e. $z_k = \sum_{n=1}^{m} i_{n+(k-1)m}$, where $1 \leq k \leq N/m$ and calculate the corresponding PDF's, $P_m(z)$. The original distribution function $P(i)$ corresponds in our new notation to $P_1(z)$. The standard deviation and kurtosis associated with $P_m(z)$ will be denoted as $\sigma(m)$ and $\kappa(m)$, respectively. We will also use the equivalent notation $P_{\Delta t}(z)$, instead of $P_m(z)$, and similarly $\sigma(\Delta t)$ and $\kappa(\Delta t)$, to emphasize the temporal scale or time horizon considered. In finance, the quantity z_k is also known as the temporal aggregation of the process.

In the simulations, we have taken $N = 2.5 \times 10^{10}$ and chosen the lattice size L sufficiently large so that the RW never touches the boundary. We have studied different sets of values (a, b), as discussed in more detail below. For instance, the set $(0.140, 0.577)$ yields $\sigma(1) \cong 0.3841$ and $\kappa(1) \cong 110$. Both, $\sigma(m)$ as well as the kurtosis $\kappa(m)$, change as a function of m. We find that $\kappa(m)$ decreases rapidly to about 48 for $m = 10$, while reaching values of the order of 3 (Gaussian behavior) for $m \approx 10^3$–10^4, as observed in real world data for long time horizons [21]. This clearly indicates that the associated PDF's are not stable; their shape depend on m and tend to the Gaussian shape for asymptotically large m. We find, in fact, that the variance $\sigma^2(m)$ behaves normally, *i.e.* $\sigma^2(m) = m\, \sigma^2(1)$.

To proceed further, we follow Mantegna and Stanley [11] and look next at the central part of the distributions, $P_m(0)$, yielding the probability that the temporal aggregation z takes the value $z = 0$ after m transactions, *i.e.* when the price of the equity at the $(n+m)$th transaction $Y_{n+m} = Y_n$. The quantity $P_m(0)$ is expected to decrease as m increases; in the language of RW it is known as the probability of return to the origin, representing the probability that after m steps the RW is back at its starting point at $z = 0$.

In the case $b = 0$, the variance of the ARCH process is constant at each transaction event ($\sigma_n^2 = a$) and $P_m(0) \sim m^{-1/2}$. For $b > 0$, the variance $\sigma_n^2 = a + b\, i_n^2$ can fluctuate considerably, since in principle the RW can reach large values of $|i_n|$. This model describes quite well the observed variations in the volatility in equity markets. The question is whether for finite b, the behavior of $P_m(0)$ may depart from its standard one as in the case of real market data. A prominent example for the latter being the PDF obtained at intervals of one minute, *i.e.* for time horizons $\Delta t_0 = 1$ min, for the Standard & Poor's 500 (S&P 500) economic index by Mantegna and Stanley [11], displaying an anomalous decay, $P_{\Delta t}(0) \sim (\Delta t)^{-0.7}$, over three orders of magnitude in Δt.

To answer the above question, we have first performed calculations for the case $a = 0.4$ and $b = 0.2 < 1/\sqrt{3}$,

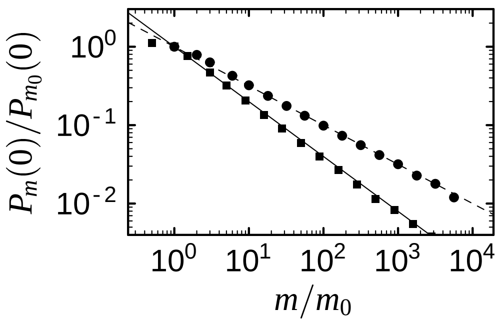

Fig. 1. Scaled plot of the probability of return to the origin $P_m(0)/P_{m_0}(0)$ *vs.* m/m_0, for discrete ARCH(1) processes with $a = 0.4$ for $b = 0.2$ (full circles) with $m_0 = 1$, and for $b = 1.2$ (full squares) with $m_0 = 2$. The straight line has the slope -0.7, the dashed line the slope -0.5 and are included as a guide.

and plotted the resulting values of $P_m(0)$ as full circles in Figure 1. The dashed line has the slope -0.5, indicating standard behavior of $P_m(0)$ in this case.

A quite different outcome emerges when the parameter b is increased further. We have considered values of b close and larger than 1, for which the second moment of the distribution diverges. The distribution remains normalizable as long as $b < 3.6377$ [19] (see also [20]). The results for $P_m(0)$ in the case $b = 1.2$ are plotted as full squares in Figure 1. Their power-law decay is consistent now with an anomalous exponent of about -0.7, over three orders of magnitude in m, in remarkable agreement with the decay observed in real data. Similar results are obtained for other values of b, but the anomalous regime with exponent -0.7 shrinks as b becomes smaller and $P_m(0)$ crosses over the standard decay at large m. The latter crossover resembles the behavior of truncated Lévy flights [11] and of ARCH processes generated with a truncated Lévy distribution (instead of the usual Gaussian one) [22].

These interesting findings for ARCH(1) processes, however, can not be easily reconciled with the real market data. In fact, for $b = 1.2$ the ARCH PDF can not fit the (1 min)-S&P 500 PDF characterized by a finite kurtosis, $\kappa \approx 30$ [11]. It is interesting to note, however, that the simple modification of the variance from the quadratic relation $\sigma_n^2 = a + b\, i_n^2$, to the linear form $\sigma_n = a' + b'\,|i_n|$, leads indeed to a sensitive improvement. Such a linear regression model may be denoted as linear ARCH(1), or LARCH(1) process. We have considered the values $a' = 0.379$ and $b' = 0.745$, for which $P_1(z)$ attains a standard deviation $\sigma(1) \cong 0.6591$ and a finite kurtosis $\kappa(1) \cong 200$. For the continuum model, the latter is finite for $b' < 3^{-1/4} = 0.75984$ [19].

For the sake of comparison with real market data, we now use the notation $P_{\Delta t}(0)$, and plot our results in the normalized form $P_{\Delta t}(0)/P_{\Delta t_0}(0)$, with $\Delta t_0 = m_0\tau$, as a function of $\Delta t/\Delta t_0 = m/m_0$. These are shown in Figure 2, together with the analysis of the Standard & Poor's 500 (S&P 500) economic index of Mantegna and Stanley [11] for which $\Delta t_0 = 1$ min.

The LARCH results for $P_{\Delta t}(0)$ at different time horizons are displayed by the full triangles in Figure 2, where

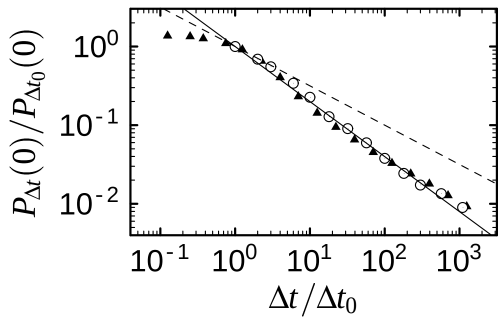

Fig. 2. Scaled plot of the probability of return to the origin $P_{\Delta t}(0)/P_{\Delta t_0}(0)$ *vs.* $\Delta t/\Delta t_0$ for a LARCH(1) process, where $\Delta t = m\tau$ and $\Delta t_0 = 1$ min. The LARCH(1) process (full triangles) was simulated for the values $a' = 0.379$ and $b' = 0.745$. Here, we have used $m_0 = 8$, *i.e.* $\tau = 7.5$ seconds, and found $\sigma(1) \cong 0.6591$. The present results are compared to the corresponding analysis of the S&P 500 data for the time horizons in the range 1 min $\leq \Delta t \leq 10^3$ min (open circles) [11]. The straight line has the slope -0.7, the dashed line the slope -0.5 and are included as a guide.

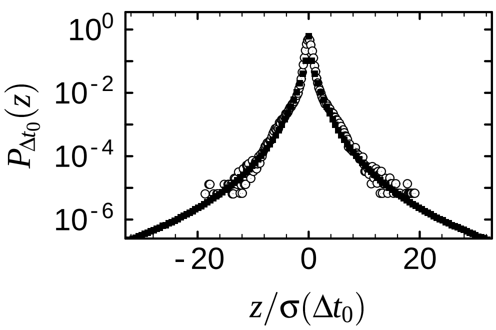

Fig. 3. The LARCH(1) PDF $P_{\Delta t_0}(z)$ *vs.* $z/\sigma(\Delta t_0)$ (full squares), for $\Delta t_0 = m_0\tau = 1$ min, with $m_0 = 8$ and $\tau = 0.125$ min, for the set $a' = 0.379$ and $b' = 0.745$, where $\sigma(\Delta t_0) \cong 0.6591\ \sqrt{8} \cong 1.8641$ and $\kappa^*(\Delta t_0) \cong 32$. For comparison, the probability distribution function of the changes of the S&P 500 index (open circles), recorded at intervals of one minute over the period from January 1984 to December 1989, is shown [11]. The reported value $\kappa^*(\Delta t_0) \cong 32$, represents an effective kurtosis obtained for values of z in the range $|z/\sigma(\Delta t_0)| \leq 18$, *i.e.* those covered by the S&P 500 data. If we take our full range of values $|z/\sigma(\Delta t_0)| \leq 500$, we find $\kappa(\Delta t_0) \cong 115$.

$m_0 = 8$ and $\tau = 7.5$ seconds. As it is apparent from the figure, the LARCH values seem to follow the empirical data over the full range of time scales up to $\Delta t \cong 10^3$ min ($m \cong 10^4$), still displaying a weak departure from the anomalous value -0.7 at larger time scales, indicating the presence of the crossover to standard decay at larger time horizons. The use of GARCH(1,1) models [3], for which $\sigma_n^2 = a + b\ x_n^2 + c\ \sigma_{n-1}^2$, may improve the fit to some extent, in particular in combination with its linear form, $\sigma_n = a + b\,|x_n| + c\,\sigma_{n-1}$. This will not be pursued further here.

In the following, we remain within the context of the LARCH(1) process and proceed by looking at the distribution function $P_{\Delta t_0}(z)$ corresponding to the time delay $\Delta t_0 = 1$ min, for the choice $a' = 0.379$ and $b' = 0.745$ used above (*cf.* Fig. 2). The results are plotted in Figure 3

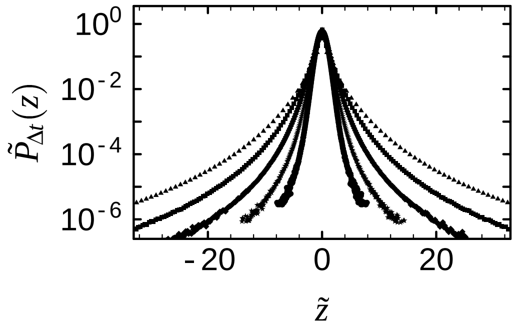

Fig. 4. Scaling plot of the LARCH(1) probability distribution function $\tilde{P}_{\Delta t}(z) \equiv P_{\Delta t}(z)/[\Delta t/\Delta t_0]^{-0.7}$ *vs.* $\tilde{z} \equiv z/[\Delta t/\Delta t_0]^{0.7}$, for several values of $\Delta t = m\tau$ ($\Delta t_0 = 1$ min), with $\tau = 0.125$ min and $m = 10$ (full triangles), 32 (full squares), 100 (full diamonds), 316 (full stars) and 1000 (full circles).

together with the S&P 500 PDF obtained at intervals of 1 min [11]. The set of parameters (a', b') employed seems appropriate since $P_{\Delta t_0}(z)$ reproduces rather well the shape of the 1 min S&P 500 PDF.

To further assess the validity of our results, we investigate the scaling behavior of the PDF's at different time horizons (*cf.* Fig. (1c) of Ref. [11]). In Figure 4, we show the scaled quantity $\tilde{P}_{\Delta t}(z) \equiv P_{\Delta t}(z)/[\Delta t/\Delta t_0]^{-0.7}$ as a function of $\tilde{z} \equiv z/[\Delta t/\Delta t_0]^{0.7}$. The data collapse found for the central parts of the PDF's for time horizons in the range 1 min $< \Delta t < 10^3$ min, corresponding to trading sequences of length $m_0 \leq m \leq 10^4$, supports the expected scaling behavior. The scaling, however, is not exact since the PDF's are not stable, the largest discrepancies are clearly observed at their tails.

Rapid Note

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